Kinematics, Dynamics and Motion Planning of Wheeled Mobile Manipulators

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*Abstract

In this paper, We first propose a systematic modeling of the wheeled mobile manipulator built from a robotic arm mounted on a wheeled mobile platform by using Lagrange dynamics equation and nonholonomic dynamics Routh equation. Then we use artificial potential field as part of feedback to accomplish the wheeled mobile manipulator's motion planning, and its stability is guaranteed by Lyapunov theory. And last, we give some results of simulation to illustrate correctness of the modeling and effectiveness of the method, and we also give rise to our next work.

Keywords: Kinematics Modeling; Dynamics Modeling; Motion Planning; Artificial Potential Field.

1. Introduction

A wheeled mobile manipulator system typically consists of a wheeled mobile base and a manipulator mounted on the mobile base. A typical characteristic of a wheeled mobile manipulator is the mobility function which is added to the basic manipulation function. Because its workspace can be enlarged considerably by moving the wheeled mobile base, and it has higher flexibility, it is very useful in certain applications such as in manufacturing, spraying paint, construction, planetary exploration and so on. However, a wheeled mobile manipulator system is a nonholonomic system because its constraint equations are not integral[1]. This makes the analysis of its kinematics, dynamics, and control more complex, and brings much more difficulty for the control and motion planning of a wheeled mobile manipulator system, since no continuous static feedback path planning scheme can be used[7][10].

Mobile manipulators have been studied during the last decade and significant work appears in literature[2]. Several papers are related to motion planning for such systems: Jaydev Desai and Vijay Kumar[3] have formulated the problem as an optimal control problem. P.Dauchez ,C.Perrier and F.Pierrot[4] minimized the total position error using a linearized model. Herbert G.Tanner and Kostas J.Kyriakopoulos[5] investigated the motion planning problem of nonholonomic mobile manipulator systems using a full state discontinuous feedback law. But these researches are less based on the systematic modeling of mobile manipulators.

In this paper, firstly, we set up the accurate modeling of kinematics and dynamics by using the Lagrange equation for the wheeled mobile manipulator system, and disregard the interior force F of the manipulator system. Secondly, we use the artificial potential field method to drive the system to move from the initial position to the destination.

This paper is organized as follows. We derive the kinematics and dynamics modeling in section 2 and 3 respectively. Section 4 examines the motion planning through the artificial potential functions. Section 5 presents some simulation results which are given to highlight correctness of the modeling and the effectiveness of the motion planning. Section 6 gives some concluding remarks.

2. Kinematics Modeling

The wheeled mobile manipulator system is shown in figure 1. The mobile platform consists of one front steering wheel and two independent rear wheels driven by two motors, and the manipulator is mounted on the center of the wheeled mobile platform. There is a motor in each joint. The link 1 can rotate around Z, and the links 2 can rotate up and down. Since the platform is driven by two driven wheels, the velocity constraints is nonholonomic(only rolling without slipping), the kinematics equation can be written as follows for point C

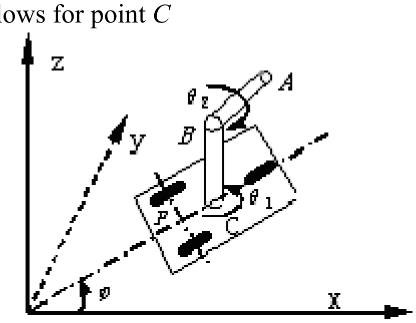

Fig1 Wheeled mobile manipulator system

$$\dot{y}\cos\varphi - \dot{x}\sin\varphi - d\dot{\varphi} = 0 \quad (1)$$

φ is the angle between the mobile platform's orientation and X-axis and d is the distance between C and P. Select $q = [x \quad y \quad \varphi \quad \theta_1 \quad \theta_2]^T$, then above equation can be express as follows:

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$$A(q)\dot{q} = 0 \tag{2}$$

Where, $A(q) = [-\sin\varphi \quad \cos\varphi \quad -d \quad 0 \quad 0]$. Let $S(q)$ be a full rank matrix (n-m) formed by a set of smooth and linearly independent vector fields spanning the null space of $A(q)$, i.e.,

$$S^T(q)A^T(q) = 0 \tag{3}$$

where

$$S(q) = \begin{bmatrix} \cos\varphi & -d\sin\varphi & 0 & 0 \\ \sin\varphi & d\cos\varphi & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

According to equation (2) and (3), we can obtain the following equation[6]

$$\dot{q} = S(q)v \tag{4}$$

where $v = [v_1 \quad v_2 \quad v_3 \quad v_4]^T$, $|v_1| \le V_{\max}$, $|v_2| \le W_{\max}$, $v_3 = \dot{\theta}_1$, and $v_4 = \dot{\theta}_2$. $V_{\max}$ and $W_{\max}$ are the maximum linear and angular velocities of the mobile platform.

Suppose the joint angle of two links respectively is θ_1, θ_2, and l_1, l_2 are the link lengths of the manipulator arms, the coordinates of the end A can be obtained from Fig1

$$\begin{aligned} x_A &= x + l_2\cos\theta_2\cos(\varphi+\theta_1) \\ y_A &= y + l_2\cos\theta_2\sin(\varphi+\theta_1) \\ z_A &= l_1 + l_2\sin\theta_2 \end{aligned} \tag{5}$$

derivate equation (4) , we can obtain

$$\begin{aligned} \dot{x}_A &= \dot{x} - l_2\sin\theta_2\cos(\varphi+\theta_1)\dot{\theta}_2 \\ &- l_2\cos\theta_2\sin(\varphi+\theta_1)(\dot{\varphi}+\dot{\theta}_1) \\ \dot{y}_A &= \dot{y} - l_2\sin\theta_2\sin(\varphi+\theta_1)\dot{\theta}_2 \\ &+ l_2\cos\theta_2\cos(\varphi+\theta_1)(\dot{\varphi}+\dot{\theta}_1) \\ \dot{z}_A &= l_2\cos\theta_2\dot{\theta}_2 \end{aligned} \tag{6}$$

Choose $\dot{X} = [\dot{x}_A \quad \dot{y}_A \quad \dot{z}_A \quad \dot{\varphi}]^T$ and $v = [v_1 \quad v_2 \quad v_3 \quad v_4]^T$, then above equation can be expressed as follows

$$\dot{X} = J(q)v \tag{7}$$

where

$$J(q) = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ J_{21} & J_{22} & J_{23} & J_{24} \\ J_{31} & J_{32} & J_{33} & J_{34} \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and

$J_{11} = \cos\varphi$, $J_{21} = \sin\varphi$

$J_{12} = -d\sin\varphi - l_2\cos\theta_2\sin(\varphi+\theta_1)$;

$J_{13} = -l_2\cos\theta_2\sin(\varphi+\theta_1)$, $J_{14} = -l_2\sin\theta_2\cos(\varphi+\theta_1)$;

$J_{22} = d\cos\varphi + l_2\cos\theta_2\cos(\varphi+\theta_1)$;

$J_{23} = l_2\cos\theta_2\cos(\varphi+\theta_1)$, $J_{24} = -l_2\sin\theta_2\sin(\varphi+\theta_2)$;

$J_{32} = 0$, $J_{33} = 0$, $J_{34} = l_2\cos\theta_2$.

3. Dynamic Modeling

A mobile manipulator system having an n-dimensional configuration space C with generalized coordinates (q_1, ….. , q_n) and subject to m constraints can be described by

$$M(q)\ddot{q} + V(q,\dot{q})\dot{q} + F(\dot{q}) + G(q) + \tau_d = B(q)\tau - A^T(q)\lambda \tag{8}$$

where $M(q) \in R^{n\times n}$ is a symmetric, positive definite inertia matrix, $V_m(q,\dot{q}) \in R^{n\times n}$ is the centripetal and coriolis matrix, $F(\dot{q}) \in R^{n\times n}$ denotes the surface friction, $G(q) \in R^n$ is the gravitational vector, τ_d denotes bounded unknown disturbances including unstructured unmodelled dynamics, $B(q) \in R^{n\times r}$ is the input transformation matrix, $\tau \in R^r$ is the input vector, $A(q) \in R^{m\times n}$ is the matrix associated with the constraints, and $\lambda \in R^m$ is the Lagrange multiplied coefficient.

There are two methods can be considered in order to derive the dynamics mathematical modeling of a wheeled mobile manipulator. One is the Newton-Euler equation, while the other is the Lagrange dynamics equation, this is the equilibrium equation of energy, it is more suitable to analyze the links' motion constrained each other. We use the Lagrange formalism to derive the dynamic equations of the wheeled mobile manipulator. Without considering the surface friction and disturbances, we can obtain the following dynamic equation

$$M(q)\ddot{q} + V_m(q,\dot{q})\dot{q} + G(q) = B(q)\tau - A^T(q)\lambda \tag{9}$$

The detail deriving is presented in appendix A. The wheeled mobile manipulator (9) have the following standard properties[5]:

1). $M(q)$ is the positive and symmetry matrix

2). $\dot{M} - 2V_m$ is skew symmetry matrix

The above properties are particularly important in the stability analysis and controllability of the wheeled mobile manipulator.

The system (9) is now transformed into a more appropriate representation for controls purposes. Substitute equation (7) into (4), we can obtain the following equation

$$\dot{q} = \tilde{S}\dot{X} \tag{10}$$

where $\tilde{S} = S(q)J^{-1}(q)$. We substitute equation (10) into (9), and then multiplying by $\tilde{S}^T$, we can eliminate the constraint item $A^T(q)\lambda$. The complete equations of motion of the wheeled mobile manipulator are given by

$$\begin{aligned} &\dot{q} = \tilde{S}\dot{X} \\ &\dot{\tilde{S}}^T M\tilde{S}\ddot{X} + \dot{\tilde{S}}^T(M\dot{\tilde{S}} + V_m\tilde{S})\dot{X} + \tilde{S}^T G = \tilde{S}^T B\tau \end{aligned} \tag{11}$$

where $X \in R^{n-m}$ is a position vector. By appropriate definitions we can rewrite equation (11) as follows:

$$\overline{M}(q)\ddot{X} + \overline{V}_m(q,\dot{q})\dot{X} + \overline{G}(q) = \overline{B}(q)\tau \tag{12}$$

where $\overline{M}(q) = \tilde{S}^T M\tilde{S}$ is a symmetric, positive definite inertia matrix, $\overline{V}_m(q,\dot{q}) = \tilde{S}^T(M\dot{\tilde{S}} + V_m\tilde{S})$ is the centripetal and coriolis matrix, $\overline{G}(q) = \tilde{S}^T G$ is the gravitational vector, $\overline{B}(q) = \tilde{S}^T B$ is the input transformation matrix, and τ is the input vector.

The system (12) is also the following properties:

1). $\overline{M}(q)$ is the positive and symmetry matrix;

2). $\dot{\overline{M}} - 2\overline{V}_m$ is skew symmetry matrix.

4. Motion planning through the artificial potential fields(APF)

Using a potential function to accomplish a certain motion implies that the trajectory of the robot is not known or calculated in advance, which means that the robot chooses autonomously its way to reach its goal[8][9][10].

4.1. General concepts

The APF approach may be described as follows. If X_d designates the goal position, the guidance of the robot with respect to n obstacles $O_i(i = 1,\cdots,n)$ can be achieved by subjecting it to the APF:

$$\varphi_{art}(X) = \varphi_{X_d}(X) + \sum_{i=1}^{n}\varphi_{O_i}(X) \tag{13}$$

where $\varphi_{art}(X)$ is the total strength of the APF at the point X, $\varphi_{X_d}(X)$ is the APF strength contribution from the attractive goal and $\varphi_{O_i}(X)$ is the contribution from the ith repulsive obstacle. This field causes the following artificial force to act on the robot:

$$F_{art}(X) = F_{X_d}(X) + \sum_{i=1}^{n}F_{O_i}(X) \tag{14}$$

where

$$F_{X_d}(X) = -\nabla\varphi_{X_d}(X),\ F_{O_i}(X) = -\nabla\varphi_{O_i}(X)$$

In this paper, we assume that there were no obstacles in the environment, so it would be a simple matter of moving in the straight line between the start and goal by the APF.

4.2. Control objective

The control objective is to develop a controller for robot dynamics given by (12) to accomplish a certain motion planning to reach the prescribed goal X_d; the trajectory of the robot is not known or calculated in advance. To accomplish this purpose we first define the position error

$$e(t) \in R^n \quad \text{as} \quad e = X_d - X \tag{15}$$

and its first derivative

$$\dot{e}(t) \in R^n \quad \text{as} \quad \dot{e} = -\dot{X} \tag{16}$$

In addition, we also define a modification PD control called PD-Plus-Gravity control as

$$\tau = \overline{B}^{-1}[\overline{G} + K_P e + K_D\dot{e}] \tag{17}$$

Using an APF and substituting $K_P e$ by F_{art} in (17) , we have

$$\tau = \hat{B}^{-1}[\hat{G} + F_{art} + K_D\dot{X}] \tag{18}$$

We know that $F_{art} = -\nabla\varphi_{art}(X)$, and (18) becomes

$$\tau = \overline{B}^{-1}[\overline{G} - \nabla\varphi_{art}(X) - K_D\dot{X}] \tag{19}$$

where $\varphi_{art}(X) = \frac{1}{2}K(X_d - X)^2$.

We can prove that the system (12) is globally asymptotically stable for the new control strategy (19). (For the proof refer Appendix B). With this feedback control, we can make the wheeled mobile manipulator move to goal position from start position.

5. Simulations

A simulation was implemented in MATLAB. Suppose the mobile manipulator move from initial position $X_s = [x_s \quad y_s \quad z_s \quad \varphi_s]$ to target $X_d = [x_d \quad y_d \quad z_d \quad \varphi_d]$, for the known φ, the rotated angle θ_1, θ_2 can be calculated from equation (5).

We suppose $X_s = [1 \quad 0 \quad 3.732 \quad 0^\circ]$, $X_d = [3 \quad 3.463 \quad 3 \quad 30^\circ]$, then we can obtain $\theta_{1s} = 0^\circ$, $\theta_{2s} = 60^\circ$ and $\theta_{1d} = 60^\circ$, $\theta_{2d} = 30^\circ$. Select $K = K_I$, $K_D = diag[2 \ 2 \ 2 \ 1]$. The designed wheeled mobile manipulator has the following parameters: $m_0 = 50kg$, $m_1 = 20kg$, $m_2 = 10kg$, $l_1 = l_2 = 2m$, $R = 0.3m$. The simulation results are given in figure 2 and figure 3 respectively. It can be seen that the present method of motion planning can drive effectively it move to destination by using the modeling of the wheeled mobile manipulator system.

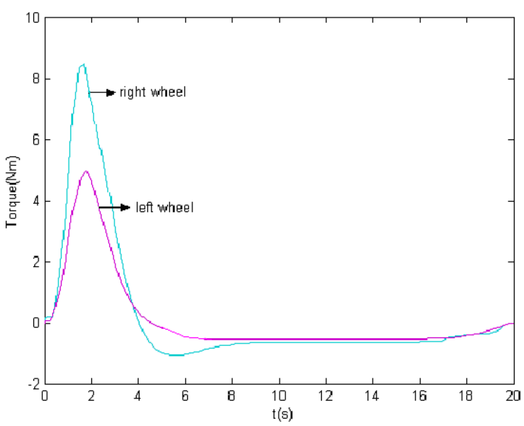

Fig 2 Torque of the right and left wheels

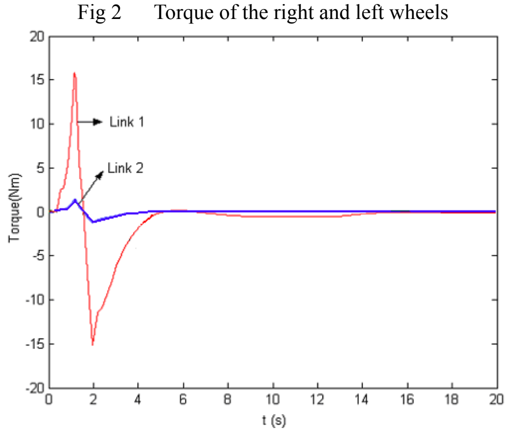

Fig3 Torque of the links

6. Conclusions

The Kinematics, Dynamics modeling of wheeled mobile manipulator and its motion planning are discussed in this paper. First, the kinematics and dynamics modeling of two link mobile manipulator are derived by using Lagrange dynamics equation and nonholonomic dynamics Routh equation. The characteristics of skew symmetry matrix $\dot{M} - 2V$ and positive symmetry matrix M guarantee the controllability and asymptotic stability of wheeled mobile manipulator system. Second, the method of artificial potential field was used to drive the wheeled mobile manipulator to accomplish the motion planning, and the results of simulation illustrated correctness of the modeling and effectiveness of the method. But we don't consider the obstacles when using artificial potential field for motion planning, and we will work over the autonomous motion planning of wheeled mobile manipulators in known environments and with unknown, possibly moving obstacles.

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Appendix A

the distance between joints and the center of mass of links is denoted as r_1, r_2 , as shown in figure 1, the coordinates of the center of mass of link 1 can be obtain

$$\begin{cases} x_1 = x \\ y_1 = y \\ z_1 = r_1 \end{cases} \tag{20}$$

the coordinates of the center of mass of link 2 can be express as follows

$$\begin{cases} x_2 = x_1 + r_2 \cos\theta_2 \cos(\varphi+\theta_1) \\ y_2 = y_1 + r_2 \cos\theta_2 \sin(\varphi+\theta_1) \\ z_2 = l_1 + r_2 \sin\theta_2 \end{cases} \tag{21}$$

the totally kinetic energy can be written as follows:

$$k = \frac{1}{2}\sum_{i=0}^{2}(m_i v_{gi}^2 + J\omega_i^2) = \frac{1}{2}m_0(\dot{x}^2 + \dot{y}^2)$$
$$+\frac{1}{2}J_0\dot{\varphi}^2 + \frac{1}{2}m_1(\dot{x}_1{}^2 + \dot{y}_1{}^2) + \frac{1}{2}J_1(\dot{\varphi}+\dot{\theta}_1)^2$$
$$+\frac{1}{2}J_2[(\dot{\varphi}+\dot{\theta}_1)^2 + \dot{\theta}_2{}^2] + \frac{1}{2}m_2(\dot{x}_2{}^2 + \dot{y}_2{}^2) \tag{22}$$

and the totally potential energy is

$$p = m_1 g r_1 + m_2 g(l_1 + r_2 \sin\theta_2) \tag{23}$$

combined equation (1) with the nonholonomic dynamics Routh equation, we can obtain

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q + A(q)^T\lambda \tag{24}$$

where Q are the forces or torques acted on the platform and links, λ is the Lagrange multiplied coefficient, the expression of the Lagrange's function is

$$L = k - p =$$
$$\frac{1}{2}m_0(\dot{x}^2+\dot{y}^2) + \frac{1}{2}J_0\dot{\varphi}^2 + \frac{1}{2}m_1(\dot{x}^2+\dot{y}^2) + \frac{1}{2}J_1(\dot{\varphi}+\dot{\theta}_1)^2$$
$$+\frac{1}{2}m_2(\dot{x}^2+\dot{y}^2) + \frac{1}{2}m_2r_2^2\sin^2\theta_2\dot{\theta}_2^2 + \frac{1}{2}m_2r_2^2\cos^2\theta_2[\dot{\varphi}+\dot{\theta}_1]^2$$
$$-m_2r_2\sin\theta_2\cos(\varphi+\theta_1)\dot{x}\dot{\theta}_2 - m_2r_2\cos\theta_2\sin(\varphi+\theta_1)\dot{x}(\dot{\varphi}+\dot{\theta}_1)$$
$$-m_2r_2\sin\theta_2\sin(\varphi+\theta_1)\dot{y}\dot{\theta}_2 + m_2r_2\cos\theta_2\cos(\varphi+\theta_1)\dot{y}(\dot{\varphi}+\dot{\theta}_1)$$
$$+\frac{1}{2}J_2[(\dot{\varphi}+\dot{\theta}_1)^2 + \dot{\theta}_2^2] - m_1gr_1 - m_2g(l_1 + r_2\sin\theta_2)$$

which results

$$\frac{\partial L}{\partial \dot{x}} = (m_0+m_1+m_2)\dot{x} - m_2r_2\sin\theta_2\cos(\varphi+\theta_1)\dot{\theta}_2$$
$$-m_2r_2\cos\theta_2\sin(\varphi+\theta_1)[\dot{\varphi}+\dot{\theta}_1]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (m_0+m_1+m_2)\ddot{x} - m_2r_2\cos\theta_2\sin(\varphi+\theta_1)[\ddot{\varphi}+\ddot{\theta}_1]$$
$$-m_2r_2\cos\theta_2\cos(\varphi+\theta_1)[\dot{\theta}_2^2 + (\dot{\varphi}+\dot{\theta}_1)^2]$$
$$+2m_2r_2\sin\theta_2\sin(\varphi+\theta_1)\dot{\theta}_2(\dot{\varphi}+\dot{\theta}_1)$$
$$-m_2r_2\sin\theta_2\cos(\varphi+\theta_1)\ddot{\theta}_2$$

$$\frac{\partial L}{\partial \dot{y}} = (m_0+m_1+m_2)\dot{y} - m_2r_2\sin\theta_2\sin(\varphi+\theta_1)\dot{\theta}_2$$
$$+m_2r_2\cos\theta_2\cos(\varphi+\theta_1)[\dot{\varphi}+\dot{\theta}_1]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = (m_0+m_1+m_2)\ddot{y} + m_2r_2\cos\theta_2\cos(\varphi+\theta_1)[\ddot{\varphi}+\ddot{\theta}_1]$$
$$-m_2r_2\cos\theta_2\sin(\varphi+\theta_1)[\dot{\theta}_2^2 + (\dot{\varphi}+\dot{\theta}_1)^2]$$
$$-2m_2r_2\sin\theta_2\cos(\varphi+\theta_1)\dot{\theta}_2(\dot{\varphi}+\dot{\theta}_1)$$
$$-m_2r_2\sin\theta_2\sin(\varphi+\theta_1)\ddot{\theta}_2$$

$$\frac{\partial L}{\partial \dot{\varphi}} = J_0\dot{\varphi} + m_2r_2^2\cos^2\theta_2[\dot{\varphi}+\dot{\theta}_1] + m_2r_2\cos\theta_2\cos(\varphi+\theta_1)\dot{y}$$
$$+J_1\dot{\varphi} + J_2(\dot{\varphi}+\dot{\theta}_1) - m_2r_2\cos\theta_2\sin(\varphi+\theta_1)\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = (J_0 + J_1 + J_2 + m_2r_2^2\cos^2\theta_2)\ddot{\varphi}$$
$$+(J_2 + m_2r_2^2\cos^2\theta_2)\ddot{\theta}_1 - m_2r_2\cos\theta_2\sin(\varphi+\theta_1)\ddot{x}$$
$$+m_2r_2\cos\theta_2\cos(\varphi+\theta_1)\ddot{y} - 2m_2r_2^2\cos\theta_2\sin\theta_2\dot{\theta}_2(\dot{\varphi}+\dot{\theta}_1)$$
$$+m_2r_2\sin\theta_2\sin(\varphi+\theta_1)\dot{x}\dot{\theta}_2 - m_2r_2\cos\theta_2\cos(\varphi+\theta_1)\dot{x}(\dot{\varphi}+\dot{\theta}_1)$$
$$-m_2r_2\sin\theta_2\cos(\varphi+\theta_1)\dot{y}\dot{\theta}_2 - m_2r_2\cos\theta_2\sin(\varphi+\theta_1)\dot{y}(\dot{\varphi}+\dot{\theta}_1)$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = (J_1+J_2)\dot{\theta}_1 + J_2\dot{\varphi} + m_2r_2^2\cos^2\theta_2[\dot{\varphi}+\dot{\theta}_1]$$
$$-m_2r_2\cos\theta_2\sin(\varphi+\theta_1)\dot{x} + m_2r_2\cos\theta_2\cos(\varphi+\theta_1)\dot{y}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = (J_1 + J_2 + m_2r_2^2\cos^2\theta_2)\ddot{\theta}_1$$
$$+(J_2 + m_2r_2^2\cos^2\theta_2)\ddot{\varphi} - m_2r_2\cos\theta_2\sin(\varphi+\theta_1)\ddot{x}$$
$$+m_2r_2\cos\theta_2\cos(\varphi+\theta_1)\ddot{y} - 2m_2r_2^2\cos\theta_2\sin\theta_2\dot{\theta}_2(\dot{\varphi}+\dot{\theta}_1)$$
$$+m_2r_2\sin\theta_2\sin(\varphi+\theta_1)\dot{x}\dot{\theta}_2 - m_2r_2\cos\theta_2\cos(\varphi+\theta_1)\dot{x}(\dot{\varphi}+\dot{\theta}_1)$$
$$-m_2r_2\sin\theta_2\cos(\varphi+\theta_1)\dot{y}\dot{\theta}_2 - m_2r_2\cos\theta_2\sin(\varphi+\theta_1)\dot{y}(\dot{\varphi}+\dot{\theta}_1)$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = J_2\dot{\theta}_2 + m_2r_2^2\sin^2\theta_2\dot{\theta}_2 - m_2r_2\sin\theta_2\cos(\varphi+\theta_1)\dot{x}$$
$$-m_2r_2\sin\theta_2\sin(\varphi+\theta_1)\dot{y}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = (J_2 + m_2r_2^2\sin^2\theta_2)\ddot{\theta}_2 - m_2r_2\sin\theta_2\cos(\varphi+\theta_1)\ddot{x}$$
$$-m_2r_2\sin\theta_2\sin(\varphi+\theta_1)\ddot{y} + 2m_2r_2^2\sin\theta_2\cos\theta_2\dot{\theta}_2^2$$
$$-m_2r_2\cos\theta_2\cos(\varphi+\theta_1)\dot{x}\dot{\theta}_2 + m_2r_2\sin\theta_2\sin(\varphi+\theta_1)\dot{x}(\dot{\varphi}+\dot{\theta}_1)$$
$$-m_2r_2\cos\theta_2\sin(\varphi+\theta_1)\dot{y}\dot{\theta}_2 - m_2r_2\sin\theta_2\cos(\varphi+\theta_1)\dot{y}(\dot{\varphi}+\dot{\theta}_1)$$

$$\frac{\partial L}{\partial x} = 0 \qquad \frac{\partial L}{\partial y} = 0$$

$$\frac{\partial L}{\partial \varphi} = m_2r_2\sin\theta_2\sin(\varphi+\theta_1)\dot{x}\dot{\theta}_2 - m_2r_2\sin\theta_2\cos(\varphi+\theta_1)\dot{\theta}_2$$
$$-m_2r_2\cos\theta_2\cos(\varphi+\theta_1)\dot{x}(\dot{\varphi}+\dot{\theta}_1)$$
$$-m_2r_2\cos\theta_2\sin(\varphi+\theta_1)\dot{y}(\dot{\varphi}+\dot{\theta}_1)$$

$$\frac{\partial L}{\partial \theta_1} = m_2 r_2 \sin\theta_2 \sin(\varphi+\theta_1)\dot{x}\dot{\theta}_2 - m_2 r_2 \sin\theta_2 \cos(\varphi+\theta_1)\dot{\theta}_2$$
$$- m_2 r_2 \cos\theta_2 \cos(\varphi+\theta_1)\dot{x}(\dot{\varphi}+\dot{\theta}_1)$$
$$- m_2 r_2 \cos\theta_2 \sin(\varphi+\theta_1)\dot{y}(\dot{\varphi}+\dot{\theta}_1)$$
$$\frac{\partial L}{\partial \theta_2} = m_2 r_2^2 \sin\theta_2 \cos\theta_2 \dot{\theta}_2^2 - m_2 r_2^2 \cos\theta_2 \sin\theta_2 [\dot{\varphi}+\dot{\theta}_1]^2$$
$$- m_2 r_2 \cos\theta_2 \cos(\varphi+\theta_1)\dot{x}\dot{\theta}_2 + m_2 r_2 \sin\theta_2 \sin(\varphi+\theta_1)\dot{x}(\dot{\varphi}+\dot{\theta}_1)$$
$$- m_2 r_2 \cos\theta_2 \sin(\varphi+\theta_1)\dot{y}\dot{\theta}_2 - m_2 r_2 \sin\theta_2 \cos(\varphi+\theta_1)\dot{y}(\dot{\varphi}+\dot{\theta}_1)$$
$$-m_2 g r_2 \cos\theta_2$$

Thus we can obtain equation (9), namely

$$M(q)\ddot{q} + V_m(q,\dot{q}) + G(q) = B(q)\tau - A(q)^T \lambda$$

where

$$M(q) = \begin{bmatrix} M_{11} & 0 & M_{13} & M_{14} & M_{15} \\ 0 & M_{22} & M_{23} & M_{24} & M_{25} \\ M_{31} & M_{32} & M_{33} & M_{34} & 0 \\ M_{41} & M_{42} & M_{43} & M_{44} & 0 \\ M_{51} & M_{52} & 0 & 0 & J_2 + m_2 r_2^2 \sin^2\theta_2 \end{bmatrix}$$

$$M_{11} = M_{22} = m_0 + m_1 + m_2$$
$$M_{13} = M_{31} = M_{14} = M_{41} = -m_2 r_2 \cos\theta_2 \sin(\varphi+\theta_1)$$
$$M_{15} = M_{51} = -m_2 r_2 \sin\theta_2 \cos(\varphi+\theta_1)$$
$$M_{23} = M_{32} = M_{24} = M_{42} = m_2 r_2 \cos\theta_2 \cos(\varphi+\theta_1)$$
$$M_{25} = M_{52} = -m_2 r_2 \sin\theta_2 \sin(\varphi+\theta_1)$$
$$M_{34} = M_{43} = J_2 + m_2 r_2^2 \cos^2\theta_2$$
$$M_{33} = J_0 + J_1 + J_2 + m_2 r_2^2 \cos^2\theta_2$$
$$M_{44} = J_1 + J_2 + m_2 r_2^2 \cos^2\theta_2$$

which is a positive symmetry matrix.

$$V_m = \begin{bmatrix} 0 & 0 & V_{13} & V_{14} & V_{15} \\ 0 & 0 & V_{23} & V_{24} & V_{25} \\ 0 & 0 & V_{33} & V_{34} & V_{35} \\ 0 & 0 & V_{43} & V_{44} & V_{45} \\ 0 & 0 & V_{53} & V_{54} & V_{55} \end{bmatrix}$$

$$V_{13} = V_{14} = -m_2 r_2 \cos\theta_2 \cos(\varphi+\theta_1)[\dot{\varphi}+\dot{\theta}_1]$$
$$+ m_2 r_2 \sin\theta_2 \sin(\varphi+\theta_1)\dot{\theta}_2$$
$$V_{15} = -m_2 r_2 \cos\theta_2 \cos(\varphi+\theta_1)\dot{\theta}_2$$
$$+ m_2 r_2 \sin\theta_2 \sin(\varphi+\theta_1)[\dot{\varphi}+\dot{\theta}_2]$$
$$V_{23} = V_{24} = -m_2 r_2 \cos\theta_2 \sin(\varphi+\theta_1)[\dot{\varphi}+\dot{\theta}_1]$$
$$- m_2 r_2 \sin\theta_2 \cos(\varphi+\theta_1)\dot{\theta}_2$$
$$V_{25} = -m_2 r_2 \sin\theta_2 \cos(\varphi+\theta_1)[\dot{\varphi}+\dot{\theta}_1]$$
$$- m_2 r_2 \cos\theta_2 \sin(\varphi+\theta_1)\dot{\theta}_2$$
$$V_{35} = V_{45} = m_2 r_2^2 \cos\theta_2 \sin\theta_2 [\dot{\varphi}+\dot{\theta}_1]$$
$$V_{33} = V_{34} = -m_2 r_2^2 \cos\theta_2 \sin\theta_2 \dot{\theta}_2 = V_{43} = V_{44}$$
$$V_{53} = V_{54} = m_2 r_2^2 \cos\theta_2 \sin\theta_2 [\dot{\varphi}+\dot{\theta}_1]$$
$$V_{55} = m_2 r_2^2 \cos\theta_2 \sin\theta_2 \dot{\theta}_2$$
$$G = \begin{bmatrix} 0 & 0 & 0 & 0 & m_2 g r_2 \cos\theta_2 \end{bmatrix}^T$$

$$B(q) = \frac{1}{r}\begin{bmatrix} \cos\varphi & \cos\varphi & 0 & 0 \\ \sin\varphi & \sin\varphi & 0 & 0 \\ R & -R & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where r is the radius of the driving wheel, and R is the distance of two wheels.

$\tau = [\tau_r \quad \tau_l \quad \tau_1 \quad \tau_2]^T$, τ_r, τ_l, τ_1, τ_2 are the driving torques of right and left wheel and two links respectively.

Appendix B (Stability of the APF controller)**.**

Consider the positive definite Lyapunov function $\hat{V}$:

$$\hat{V} = \varphi_{art}(X) + \frac{1}{2}\dot{X}^T \overline{M}\dot{X}$$

where $\varphi_{art}(X) > 0$ for all $X \neq X_d$. The time derivative of $\hat{V}$ is given by:

$$\dot{\hat{V}} = \dot{X}^T \nabla\varphi_{art}(X) + \frac{1}{2}\dot{X}^T \dot{\overline{M}}\dot{X} + \dot{X}^T \overline{M}\ddot{X}$$
$$= \frac{1}{2}\dot{X}^T [\dot{\overline{M}} - 2\overline{V}_m]\dot{X} - \dot{X}^T K_D \dot{X}$$

$$\therefore\ \dot{\hat{V}} = -\dot{X}^T K_D \dot{X}$$

K_D is positive definite that $\dot{\hat{V}} \leq 0$. The system is globally asymptotically stable.