Natural Resolution of Ill-posed Inverse Kinematics for Redundant Robots: A Challenge to Bernstein’s Degrees-of-Freedom Problem

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Abstract—This paper aims at challenging Bernstein’s problem called the “Degrees-of-Freedom problem”, which is known to remain unsolved from both the physiological and the robotics viewpoints. More than a half century ago A.N. Bernstein observed and claimed that “dexterity” resident in human limb motion emerges from involvement of multi-joint movements with surplus DOF. It is also said in robotics that redundancy of DOFs in robot mechanisms may contribute to enhancement of dexterity and versatility. However, kinematic redundancy incurs a problem of ill-posedness of inverse kinematics from task-description space to joint space. In the history of robotics research such ill-posedness problem of inverse-kinematics has not yet been attacked directly but circumvented by introducing an artificial performance index and determining uniquely an inverse kinematics solution by minimizing it. Instead of it, this paper introduces two novel concepts named “stability on a manifold” and “transferability to a submanifold” in treating the case of human multi-joint movements of reaching and shows that a sensory feedback from task space to joint space together with a set of adequate dampings and damping coefficients. It is also shown that these novel concepts can cope with annoying characteristics called “variability” of redundant joint motions seen typically in human skilled reaching.

KEY WORDS
Redundant Robots, Multi-joint Reaching, Ill-posedness of Inverse Kinematics, Bernstein’s Problem, Surplus DOF

I. INTRODUCTION

This paper is concerned with a challenge to one of unsolved problems posed by A.N. Bernstein [1] [2] as the Degrees-of-Freedom problem, particularly, in case of human or robotic multi-joint movements of reaching as shown in Fig.1. The problem is how to generate a joint motion so as to transfer the endpoint of an upper limb with four joints (shoulder, elbow, wrist, and finger MP joint) to a given target point \( x_d = (x_d, y_d) \) in the two-dimensional horizontal plane. Since the objective task \( x_d \) is given in the task space \( x \in X(=R^2) \) and the joint coordinates \( q = (q_1, q_2, q_3, q_4)^T \) are of four-dimension, there exists an infinite number of inverses \( q_d \) that realize \( x(q_d) = x_d \) and hence the problem of obtaining inverse kinematics from the task description space \( X \) to the 4-dimensional joint space becomes ill-posed. Under this circumstances, however, it is necessary to generate joint motions \( q(t) \) starting from a given initial point \( x(0) = (x(0), y(0)) \) in \( X \) with some initial posture \( q(0) = (q_1(0), \ldots, q_4(0))^T \) and leading the endpoint trajectory \( x(t) \) to reach the target \( x_d \) as \( t \to \infty \). In order to get rid of such ill-posedness, many methods have been proposed as surveyed in a special issue of the journal [3] and a book specially dedicated to problems of redundancy [4]. Most of them are based on an idea of introducing some extra and artificial performance index for determining uniquely an appropriate joint space trajectory by minimizing it. In fact, examples of such performance index in robotics research are the followings: kinetic energy [5], quadratic norm of joint control torque [6], manipulability index [7], virtual fatigue function [8], etc. [9] [10]. Most of proposed methods have been explicitly or implicitly based on the Jacobian pseudoinverse approach for planning an optimized joint velocity trajectory \( \dot{q}(t) \) together with an extra term \( (I - J^+(q)J(q))\psi \), where \( \psi \) is determined by optimizing the performance index, \( J(q) \) stands for the Jacobian matrix of task coordinates \( x \) in joint coordinates \( q \), and \( J^+(q) \) the pseudoinverse of \( J(q) \). In the history of robot control the idea of use of the pseudoinverse for generation of joint trajectories for redundant robots was initiated by Whitney [11] and analyzed more in details by Liegeois [12]. However, it is impossible to calculate \( J^+(q_d) \) in advance because \( q_d \) is...
undetermined. Therefore, it is recommended that a control signal $u$ to be exerted through joint actuators of the robot is designed as

$$u = -C\dot{q} - J^T(q)k\Delta x$$  \hspace{1cm} (2)$$

where $\Delta x = x - x_d$, $k$ a stiffness parameter. This means that the control signal is composed of only two terms, one is a damping term (angular velocity feedback) and the other is a sensory feedback from task space with the stiffness parameter $k$ modified by the transpose of Jacobian $J(q)$. This is nothing else but a task-space PD feedback scheme with damping shaping in the case of control of non-redundant robot manipulators [27]. It is proved theoretically that adequate choices of gain matrices $C$ and stiffness parameter $k$ render the closed-loop system dynamics convergent as time elapses, that is, $x(t) \rightarrow x_d$ and $\dot{q}(t) \rightarrow 0$ as $t \rightarrow \infty$. However, owing to the joint redundancy, the convergence in task space does not directly imply the convergence of joint variables $q(t)$ to some posture. In the paper, by introducing a novel concept named “stability on a manifold” it is shown that $q(t)$ remains in a specified region in joint space such that the Jacobian matrix is nondegenerated and there does not arise any unexpected self-motion inherent to redundant systems. In other words, this concept suggests that the problem of elimination of joint redundancy need not be solved but can be ignored in control of the dynamics. Or it can be said that a kind of physical principle for economies of skilled motions may work in elimination of redundancy. Another concept named “transferability to a submanifold” is also introduced for discussing the asymptotic convergence in a case of middle-range reaching. These two concepts were originally and very recently defined in cases of control of multi-fingered hands with joint redundancy and control of a hand-writing robot with surplus DOFs [28][29]. Finally, the characteristics of human skilled multi-joint reaching called “variability” in joint motion raised by Latash [30] is interpreted by means of these two novel concepts.

II. CLOSED-LOOP DYNAMICS OF MULTI-JOINT REACHING MOVEMENT

Lagarde’s equation of motion of a multi-joint system whose motion is confined to a plane as shown in Fig. 1 is described by the formula (see [30])

$$H(q)\ddot{q} + \left\{ \frac{1}{2}H(q) + S(q, \dot{q}) \right\} \dot{q} = u$$  \hspace{1cm} (3)$$

where $q = (q_1, q_2, q_3)^T$ denotes the vector of joint angles, $H(q)$ the inertia matrix, and $S(q, \dot{q})\dot{q}$ the gyroscopic force term including centrifugal and Coriolis force. It is well known that the inertia matrix $H(q)$ is symmetric and positive definite and there exist a positive constant $h_m$ together with a positive definite constant diagonal matrix $H_0$ such that

$$h_mH_0 \leq H(q) \leq H_0$$  \hspace{1cm} (4)$$

for any $q$. It should be also noted that $S(q, \dot{q})$ is skew symmetric and linear and homogeneous in $\dot{q}$. Any entry of $H(q)$ and $S(q, \dot{q})$ is constant or a sinusoidal function of components of $q$.

For a given specified target position $x_d = (x_d, y_d)$ as shown in Fig. 1, if the control input of eq.(2) is used at joint actuators then the closed-loop equation of motion of the system can be expressed as

$$H(q)\ddot{q} + \left\{ \frac{1}{2}H(q) + S(q, \dot{q}) + C \right\} \dot{q} + J^T(q)k\Delta x = 0$$  \hspace{1cm} (5)$$

which follows from substitution of eq.(2) into eq.(3). Since $\dot{x} = J(q)\dot{q}$, the inner product of eq.(5) with $\dot{q}$ is reduced to

$$\frac{d}{dt}E = -\dot{q}^TC\dot{q}$$  \hspace{1cm} (6)$$
If for an arbitrarily given
only a quadrative term of two-dimensional vector
the kinetic energy as a positive definite quadratic function of
of 2-dimension defined as
noted that the scalar function
analysis, since it shows that the derivative of a scalar function
dynamics of eq.(5). It also reminds us of Lyapunov’s stability
[31]), the relation of eq.(6) denotes passivity of the closed-loop
in cartesian space. As it is well known in robot control (see
where
stands for the total energy, i.e.,

\[ E(q, \dot{q}) = \frac{1}{2} \dot{q}^T H(q) \dot{q} + \frac{k}{2} \Delta x^T \Delta x \quad (7) \]

Evidently the first term of this quantity \( E \) stands for the kinetic energy of the system. The second term is called an artificial potential in this paper that appears due to addition of equilibrium point control \( J^T(q) \dot{x} \) based on the error \( \Delta x \) expressed in cartesian space. As it is well known in robot control (see
[31]), the relation of eq.(6) denotes passivity of the closed-loop dynamics of eq.(5). It also reminds us of Lyapunov’s stability analysis, since it shows that the derivative of a scalar function \( E \) in time \( t \) is negative semi-definite. However, it should be noted that the scalar function \( E(q, \dot{q}) \) is not positive definite with respect to the state vector \( \{q, \dot{q}\} \in R^8 \). In fact, \( E \) includes only a quadrative term of two-dimensional vector \( \Delta x \) except the kinetic energy as a positive definite quadratic function of \( q \). Therefore, it is natural and reasonable to introduce a manifold of 2-dimension defined as

\[ M_2 = \{ (q, \dot{q}) : E(q, \dot{q}) = 0 \} \]

which is called the zero space in the literature of robotics research (for example, [6]). Next, consider a posture \((q^0, 0)\) with still state (i.e., \( \dot{q} = 0 \)) whose endpoint is located at \( x_d \), i.e., \( x(q^0) = x_d \) and hence \((q^0, 0) \in M_2 \), and analyze stability of motion of the closed-loop dynamics in a neighborhood of this equilibrium state. This equilibrium state in \( R^8 \) is called in this paper the reference equilibrium state.

It is now necessary to introduce the concept of neighborhoods of the reference equilibrium state \((q^0, 0) \in M_2 \) in \( R^8 \), which are conveniently defined with positive parameters \( \delta > 0 \) and \( r_0 > 0 \) as

\[ N^8(\delta, r_0) = \{ (q, \dot{q}) : E(q, \dot{q}) \leq \delta^2 \quad \text{and} \quad ||q - q^0||_K \leq r_0 \} \]

where \( ||q - q^0||_K = \{\frac{1}{2} (q - q^0)^T H(q)(q - q^0)\}^{1/2} \) (see Fig.4). The necessity of imposing the inequality condition \( ||q - q^0||_K \leq r_0 \) comes from avoiding arise of possible movements such as self-motion [32] due to redundancy of DOFs far from the original posture. In fact, for the given endpoint \( x_d \) with the reference state as shown by the mark \( F \) in Fig.3, one possible state with the posture marked by \( S \) in Fig.3 may be inside \( N^8(\delta, r_0) \) but another state marked by \( U \) must be excluded from the neighborhood \( N^8(\delta, r_0) \) by choosing \( r_0 > 0 \) appropriately, because the overall posture of \( U \) is by far deviated from that of the original reference equilibrium state \((q^0, 0)\). Further, it is necessary to assume that the reference equilibrium state \((q^0, 0)\) is considerably distant from the posture that has singularity of Jacobian matrix \( J(q) \), which happens if and only if \( q_2 = q_3 = q_4 = 0 \).

We are now in a position to define the concept of stability of the reference equilibrium state lying on the manifold \( M_2 \).

**Definition 1** If for an arbitrarily given \( \varepsilon > 0 \) there exist a constant \( \delta > 0 \) depending on \( \varepsilon \) and another constant \( r_1 > 0 \) independent of \( \varepsilon \) and less than \( r_0 \) such that a solution trajectory \((q(t), \dot{q}(t))\) of the closed-loop dynamics of eq.(5) starting from any initial state \((q(0), \dot{q}(0))\) inside \( N^8(\delta(\varepsilon, r_1)) \) remains in \( N^8(\varepsilon, r_0) \), then the reference equilibrium state \((q^0, 0)\) is called stable on a manifold (see Fig.4).
If for a reference equilibrium state \((q^0, 0) \in R^8\) there exist constants \(\varepsilon_1 > 0 \) and \(r_1 > 0 \) \((r_1 < r_0)\) such that any solution of the closed-loop dynamics of eq.(5) starting from an arbitrary initial state in \(N^8(\varepsilon_1, r_1)\) remains in \(N^8(\varepsilon_1, r_0)\) and converges asymptotically as \(t \to \infty\) to some point on \(M_2 \cap N^8(\varepsilon_1, r_0)\), then the neighborhood \(N^8(\varepsilon_1, r_1)\) of the reference equilibrium state \((q^0, 0)\) is said to be transferable to a submanifold of \(M_2\).

This definition means that, even if a still state \((q^0, 0) \in M_2\) of the multi-joint system is forced to move instantly to a different state \((q(0), \dot{q}(0))\) in a neighborhood of \((q^0, 0)\) by being exerted from some external disturbance, the sensory feedback control of eq.(2) assures that the system’s state soon starting from an arbitrary initial state in \(\mathbb{R}^8\) such that any solution of the closed-loop dynamics of eq.(5) eventually converges to the target if the stiffness \(k\) is chosen adequately. For example, we show several endpoint trajectories for different \(k\) in Fig.5 obtained by computer simulation based on a human model shown in Table I in the case that the initial point \(x(0)\) in task space \(X(= R^2)\) and its corresponding initial posture \(q(0)\) are set as in Table II, where damping gains are chosen as

\[
c_1 = c_2 = c_3 = c_4 = 0.0025 \quad [\text{Ns}] \quad (8)
\]

The endpoint trajectory \(x(t)\) starting from the initial posture shown as S) in Fig.3 is going to approach the target but overruns and oscillates around the target \(x_d\), but in a long run it converges to the target and stops with the final posture shown as U) in Fig.3 when \(k = 10.0\) is chosen.

### Table I
LENGTHS OF UPPER ARM \(l_1\), LOWER ARM \(l_2\), PALM \(l_3\), AND INDEX FINGER \(l_4\) TOGETHER WITH CORRESPONDING LINK MASSES AND INERTIAS. THE DATA ARE TAKEN FROM A 5 YEARS OLD CHILD.

<table>
<thead>
<tr>
<th>Arm</th>
<th>link1 length</th>
<th>(l_1)</th>
<th>(0.175) [m]</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>link2 length</td>
<td>(l_2)</td>
<td>(0.170) [m]</td>
<td>(3)</td>
</tr>
<tr>
<td></td>
<td>link3 length</td>
<td>(l_3)</td>
<td>(0.000) [m]</td>
<td>(3)</td>
</tr>
<tr>
<td></td>
<td>link4 length</td>
<td>(l_4)</td>
<td>(0.000) [m]</td>
<td>(3)</td>
</tr>
<tr>
<td></td>
<td>link1 cylinder radius</td>
<td>(r_1)</td>
<td>(0.024) [m]</td>
<td>(3)</td>
</tr>
<tr>
<td></td>
<td>link2 cylinder radius</td>
<td>(r_2)</td>
<td>(0.023) [m]</td>
<td>(3)</td>
</tr>
<tr>
<td></td>
<td>link3 cylinder depth</td>
<td>(d_3)</td>
<td>(0.021) [m]</td>
<td>(3)</td>
</tr>
<tr>
<td></td>
<td>link4 cylinder radius</td>
<td>(r_4)</td>
<td>(0.0009) [m]</td>
<td>(3)</td>
</tr>
<tr>
<td></td>
<td>link1 mass</td>
<td>(m_1)</td>
<td>(0.325) [kg]</td>
<td>(3)</td>
</tr>
<tr>
<td></td>
<td>link2 mass</td>
<td>(m_2)</td>
<td>(0.266) [kg]</td>
<td>(3)</td>
</tr>
<tr>
<td></td>
<td>link3 mass</td>
<td>(m_3)</td>
<td>(0.0756) [kg]</td>
<td>(3)</td>
</tr>
<tr>
<td></td>
<td>link4 mass</td>
<td>(m_4)</td>
<td>(0.0388) [kg]</td>
<td>(3)</td>
</tr>
<tr>
<td></td>
<td>link1 inertia moment</td>
<td>(I_1)</td>
<td>(9.07 \times 10^{-4}) [kgm²]</td>
<td>(3)</td>
</tr>
<tr>
<td></td>
<td>link2 inertia moment</td>
<td>(I_2)</td>
<td>(6.73 \times 10^{-4}) [kgm²]</td>
<td>(3)</td>
</tr>
<tr>
<td></td>
<td>link3 inertia moment</td>
<td>(I_3)</td>
<td>(2.56 \times 10^{-4}) [kgm²]</td>
<td>(3)</td>
</tr>
<tr>
<td></td>
<td>link4 inertia moment</td>
<td>(I_4)</td>
<td>(1.80 \times 10^{-4}) [kgm²]</td>
<td>(3)</td>
</tr>
</tbody>
</table>

### Table II
INITIAL CONDITIONS

| \(q(0)\) | \(70.0\) [deg] | \(3\) |
| \(q_1(0)\) | \(50.0\) [deg] | \(3\) |
| \(q_2(0)\) | \(30.0\) [deg] | \(3\) |
| \(q_3(0)\) | \(80.0\) [deg] | \(3\) |
| \(q(0)\) | \(-0.1157\) [m] | \(3\) |
| \(q_1(0)\) | \(0.2597\) [m] | \(3\) |

### Table III
NUMERICAL VALUES OF INERTIA MATRIX \(H(q(t))\) AT \(t = 0\) WHOSE POSTURE IS EXPRESSED IN FIG.2

\[
H(0) = \begin{bmatrix}
2.812610e-02 & 9.906791e-03 & 5.4228169e-04 & -2.5192563e-05 \\
9.906791e-03 & 5.7745657e-03 & 4.8854896e-04 & -1.1000662e-06 \\
5.4228169e-04 & 4.8854896e-04 & 1.2002931e-04 & 7.4183440e-06 \\
-2.5192563e-05 & -1.1000662e-06 & 7.4183440e-06 & 5.8919051e-06
\end{bmatrix}
\]

### Fig. 5
Endpoint trajectories of multi-joint reaching movements when damping factors of eq.(8) is used.

### III. MIDDLE-RANGE REACHING

According to the energy balance law expressed by eq.(6), the total energy \(E(t) = E(q(t)), \dot{q}(t)\) is decreasing with increasing \(t\) as far as \(\dot{q} \neq 0\) for an arbitrary positive definite damping gain matrix \(C\). However, the motion profile \(x(t)\) of the endpoint is quite sensitive to choice for \(c_i > 0\) for \(i = 1, \cdots, 4\) though for a broad range of choice for \(c_i\) \((i = 1, \cdots, 4)\) the endpoint \(x(t)\) eventually converges to the target if the stiffness \(k\) is chosen adequately. For example, we show several endpoint trajectories for different \(k\) in Fig.5 obtained by computer simulation based on a human model shown in Table I in the case that the initial point \(x(0)\) in task space \(X(= R^2)\) and its corresponding initial posture \(q(0)\) are set as in Table II, where damping gains are chosen as

\[
c_1 = c_2 = c_3 = c_4 = 0.0025 \quad [\text{Ns}] \quad (8)
\]

The endpoint trajectory \(x(t)\) starting from the initial posture shown as S) in Fig.3 is going to approach the target but overruns and oscillates around the target \(x_d\), but in a long run it converges to the target and stops with the final posture shown as U) in Fig.3 when \(k = 10.0\) is chosen.

### Fig. 6
Endpoint trajectories of reaching movements when damping factors of eq.(9) is used.
Then, numerical solutions of the closed-loop dynamics of eq.(5) for different stiffness parameters give rise to transient responses of the endpoint trajectory \(x(t), y(t)\) in Fig.6, \(x(t)\) and \(y(t)\) in Fig.7 and \(q_1(t)\) and \(q_4(t)\) in Fig.8. As shown in Figs.6 and 7, the endpoint trajectories become well approximately rectilinear and do not change much for different stiffness parameters, though the speed of convergence to the target depends on \(k\). The best choice of \(k\) in this chosen set of damping factors given in eq.(9) must be around \(k = 10.0\) [N/m].

Now let us discuss how to select such a good set of damping factors as in eq.(9). If one of the tightest (smallest) diagonal matrix \(H_0\) satisfying eq.(4) is found, then it is possible to select \(C\) in such a way that

\[
C \geq 3.0 H_0^{1/2}
\]  

Further, it should be noted that such a matrix \(H_0\) can be selected as the smallest constant diagonal matrix satisfying

\[
H(q) \leq H_0 \quad \text{for all } q \text{ such that } \|x(q) - q_d\| \leq r
\]  

where \(r\) denotes the euclidean distance between the starting endpoint \(x(0) = (x(0), y(0))\) and the target \(q_d\), because according to eq.(6) the endpoint should remain inside the circle \(\|x(t) - q_d\| \leq r\) for any \(t > 0\). In the case of middle-range reaching with \(r = 12.76\) [cm] for a typical five-years old child with 1.07 [m] in height, the initial value of \(H(q)\) with the posture shown in Fig.2 is evaluated as in Table III. We evaluate a tighter bound \(H_0\) starting from the data of \(H(0)\) in Table III by adding each possible contribution of off-diagonal element \(|H_{ij}(0)|\) to \(H_i(0)\) and \(H_{ij}(0)\). By the same computer simulation based on Table I and II, we find that such a choice of \(C = 3.0 H_0^{1/2}\) as numerically given in eq.(9) gives rise to

\[
9.0 C^{-1} H(q) C^{-1} < I_k
\]  

during the transient process of reaching as shown in Fig.9.

IV. STABILITY ON A MANIFOLD AND TRANSFERABILITY

Now it is possible to show that a reference still state \((q^0, 0)\) shown by the posture F in Fig.3 is stable on a manifold and there does not arise any unreasonable self-motion in the vicinity of \((q^0, 0)\) if \(q(0)\) is close to \(q^0\) and damping factors are chosen as in eq.(9). Then, it is possible to show that taking inner product between eq.(5) and \(\{\dot{q} + C^{-1} kJ^T \Delta x\}\) yields

\[
\frac{d}{dt} W(k; \Delta x, \dot{q}) + k h(\Delta x, \dot{q})
\]

\[
= -\dot{q}^T C \dot{q} - k^2 \Delta x^T J C^{-1} J^T \Delta x
\]  

where

\[
h(\Delta x, \dot{q}) = \Delta x^T J C^{-1} \left( \frac{1}{2} \dot{H} - S \right) \dot{q}
\]

\[
- \Delta x^T J C^{-1} H \dot{q} - \dot{q}^T J^T J C^{-1} H \dot{q}
\]  

and

\[
W(k; \Delta x, \dot{q}) = \frac{1}{2} \dot{q}^T \dot{q} + k C^{-1} J^T \Delta x H(q + k C^{-1} J^T \Delta x)
\]

\[
+ \frac{1}{2} k \Delta x^T \left( 2 I - k J C^{-1} H C^{-1} J^T \right) \Delta x
\]  

According to Latash [30], human skilled multi-joint reaching is characterized as follows:

a) The profile of the endpoint trajectory in task space \(X\) becomes closely rectilinear,

b) the velocity profile of it in \(X\) becomes symmetric and bell-shaped,

c) the acceleration profile has double peaks,

d) but each profile of time histories of joint angles \(q_i(t)\) and angular velocities \(\dot{q}_i(t)\) may differ for \(i = 1, 2, \cdots, 4\).

We are now in a position to answer the question whether it is possible to find a set of adequate damping factors \(c_i\) \((i = 1, 2, \cdots, 4)\) together with an adequate stiffness parameter \(k > 0\) so that the simpler sensory feedback of eq.(2) leads to the skilled motion of reaching realizing an approximately rectilinear endpoint trajectory without incurring any noteworthy self-motion. We select damping factors as follows:

\[
c_1 = 0.69, \quad c_2 = 0.375, \quad c_3 = 0.096, \quad c_4 = 0.024
\]  

Fig.7. Transient responses of \(x\) and \(y\) corresponding to motions shown in Fig.6.

Fig.8. Transient responses of \(q_1\) and \(q_4\) corresponding to motions shown in Fig.6.

Fig.9. Transient behaviors of singular values of \(9.0 C^{-1} H(q(t)) C^{-1}\) and eigenvalues of \(J(q(t)) C^{-1} J(q(t))\) when damping factors of eq.(9) is used.
Equation (13) can be rewritten into
\[
\frac{d}{dt}W(k; \Delta x, \dot{q}) = -\dot{q}^T C \dot{q} - k^2 \Delta x^T J C^{-1} J^T \Delta x - kh(\Delta x, \dot{q}) \\
\leq -\gamma W(k; \Delta x, \dot{q}) - kh(\Delta x, \dot{q}) - f(k; \Delta x, \dot{q})
\] (16)
where
\[
f(k; \Delta x, \dot{q}) = \dot{q}^T \left( C - \frac{\gamma}{2} H - \frac{\gamma k}{4} H \right) \dot{q} \\
+k \Delta x^T (k J C^{-1} J^T \gamma \Delta x - \gamma J C^{-1} H C^{-1} J^T) \Delta x
\] (17)
and the following inequality is used:
\[
\Delta x^T J C^{-1} H \dot{q} \leq \Delta x^T J C^{-1} H C^{-1} J^T \Delta x + \frac{1}{4} \dot{q}^T H \dot{q}
\] (18)
Now, let us choose
\[
k = 10.0, \quad \gamma = 2.0
\] (19)
and note that
\[
\dot{q}^T \left( C - \frac{\gamma}{2} H - \frac{\gamma k}{4} H \right) \dot{q} \geq \frac{1}{2} \dot{q}^T C \dot{q}
\] (20)
Since \( C \) satisfies inequalities (10), (12) and \( J(q)C^{-1}J^T(q) \geq 0.215 I_2 \) for all \( t \geq 0 \) as shown in Fig.9 too and the spectral radius of \( J(q)J^T(q) \) is apparently smaller than 1/4 according to Table 1, it follows that
\[
\Delta x^T (k J C^{-1} J^T \gamma \Delta x - \gamma J C^{-1} H C^{-1} J^T) \Delta x \\
\geq \Delta x^T (2.15 I_2 - 2.0 I_2 - (2/9) J J^T) \Delta x \geq 0
\] (21)
Finally, it is possible to conclude that
\[
-k \dot{h}(\Delta x, \dot{q}) - f(k; \Delta x, \dot{q}) \leq -\frac{1}{2} \dot{q}^T C \dot{q} - 10.0 h(\Delta x, \dot{q}) \leq 0
\] (22)
(the proof is omitted in this paper). In conclusion, by substituting eq.(22) into eq.(16) we obtain
\[
\frac{d}{dt}W(\Delta x, \dot{q}) \leq -2.0 W(\Delta x, \dot{q})
\] (23)
where we denote \( W(k; \Delta x, \dot{q}) = W(\Delta x, \dot{q}) \). Apparently, eq.(23) leads to
\[
W(t) = W(\Delta x(t), \dot{q}(t)) \leq e^{-2t} W(\Delta x(0), \dot{q}(0)) \\
\leq e^{-2t} k \| \Delta x(0) \|^2
\] (24)
Since it is possible to ascertain that \( W(\Delta x, \dot{q}) \geq (13k/14) \| \Delta x \|^2 \) (the proof is omitted), we conclude that
\[
\| \Delta x(t) \|^2 \leq (14/13) \| \Delta x(0) \|^2 e^{-2t}
\] (25)
Now it is possible to evaluate the bound for \( \| q(t) - q(0) \|_K \) on the basis of eq.(5), which by taking integral can be reduced to
\[
C(q(t) - q(0)) = -H(q(t)) \dot{q}(t) + H(q(0)) \dot{q}(0) \\
+ \int_0^t \left( \frac{1}{2} \dot{H} - S \right) \dot{q} d\tau - \int_0^t J^T \Delta x d\tau
\] (26)
Since \( C \geq 3.0 H(q)^{1/2} J(q)J^T(q) \leq (1/4) I_2 \), and the spectral radius of \( H(q) \) is less than 0.05 as predicted from Table III, it follows that
\[
\| q(t) - q(0) \|_K \leq \frac{1}{3 \sqrt{2}} H(t(q)) \| q(t) - q(0) \| \\
\leq \frac{1}{3 \sqrt{2}} \left( H(t(q)) \| q(t) - q(0) \| + H(q(0)) \| q(t) - q(0) \| \right) \\
+ \frac{\beta}{3 \sqrt{2}} \int_0^t \dot{q}^T C \dot{q} d\tau + \frac{10}{3 \sqrt{2}} \int_0^t \frac{1}{2} \| \Delta x \| d\tau
\] (27)
where \( \beta \) can be evaluated as being less than 0.02. Hence,
\[
\| q(t) - q(0) \|_K \leq \frac{1}{3 \sqrt{2}} \frac{1}{\sqrt{10}} \left( \sqrt{K(t)} + \sqrt{K(0)} \right) \\
+ \frac{0.02}{2 \sqrt{2}} E(0) + \frac{5}{3 \sqrt{2}} \int_0^t \frac{14}{13} \| \Delta x(0) \| e^{-\tau} d\tau \\
\leq \frac{1}{2 \sqrt{2}} E(0) + \frac{0.01}{2 \sqrt{2}} E(0)
\] (28)
where inequalities \( K(t) \leq E(t) \leq E(0) \) are used.
Now, assume that a reference still state \( (q^0, 0) \) with \( x(q^0) = x_d \) is given as shown in Fig.3 with the posture \( F ) \) and choose positive numbers \( \delta_1 > 0 \) and \( r_0 > 0 \) appropriately so that any \( q \) in \( N^8(\delta_1, r_0) \) satisfies eq.(11), eq.(12), and another inequality \( J(q)C^{-1}J^T(q) \geq 0.215 I_2 \). Then, it is possible to prove that

**Theorem 1.** The reference state \( (q^0, 0) \) is stable on a manifold for the closed-loop dynamics of eq.(5) with \( k = 10.0 \) and damping factors given in eq.(9).

**Proof.** For an arbitrary given \( \varepsilon > 0 \), choose \( r_1 = r_0/2 \) and \( \delta(\varepsilon) = \min(\varepsilon, \delta_1, r_1) \), and denote the solution \( (q(t), \dot{q}(t)) \) to eq.(5) starting from \( (q(0), \dot{q}(0)) \) lying in \( N^8(\delta(\varepsilon), r_1) \). Without loss of generality, we assume that all \( \varepsilon, \delta_1, \) and \( r_1 \) are less than 1.0. Then, according to eq.(28), it follows that
\[
\| q(t) - q^0 \|_K \leq \| q(t) - q^0 \|_K + \| q(t) - q(0) \|_K \\
\leq r_1 + \left( \frac{1}{2} E(0) + \frac{0.01}{\sqrt{2}} E(0) \right) \leq r_0 \\
+ \left( \frac{1}{2} \delta(\varepsilon) + \frac{1}{4} \delta^2(\varepsilon) \right) \leq r_0 + r_0 \left( \frac{1}{2} + \frac{1}{4} \right) < r_0
\] (29)
which proves that \( (q(t), \dot{q}(t)) \) remains in \( N^8(\varepsilon, r_0) \).

Clearly the following theorem is valid.

**Theorem 2.** The neighborhood \( N^8(\delta_1, r_1) \) is transferable to a submanifold of \( M_2 \).

The transferability to a submanifold of \( M_2 \) is valid for not only a neighborhood \( N^8(\delta_1, r_1) \) of the reference still state \( (q^0, 0) \) but also a neighborhood of the starting posture \( S \) of Fig.3 or Fig.2 in the case of middle-range reaching as shown in Figs.6 to 8, because throughout the way to the target inequalities (12) and \( JC^{-1}J^T > 0.215 I_2 \) are satisfied as shown in Fig.9 and thereby inequalities (24) and (25) are valid. However, it is necessary to show that, even in the case of middle-range reaching, there does not arise any unreasonable self-motion like the posture \( U \) shown in Fig.3 if a proper set of damping factors is chosen. In the case of damping factors of eq.(9) with the chosen stiffness \( k = 10.0 \), we evaluate an upper-bound on each magnitude of change of joint angle,
\[ q_i(\infty) - q_i(0) \]. For example, consider the case of \( i = 3 \). First note that if \( J(q) \) is decomposed to \( J(q) = (J_1, J_2, J_3, J_4) \) then \( J_3^T = (l_3s_123 + l_4s_1234, -l_3c_123 + l_4c_1234) \), where we denote \( \sin(q_1 + q_2 + q_3) \) by \( s_123 \) and \( \cos(q_1 + q_2 + q_3 + q_4) \) by \( c_{123} \) and so on. Therefore, \( ||J_3|| \leq l_3 + l_4 = 0.12 \) [m] and from eq.(26) it follows that

\[ c_3|q_3(\infty) - q_3(0)| \leq \beta \int_0^\infty \dot{q}^T C \dot{q} dt + k \int_0^\infty ||J|| \| \Delta x \| dt \]
\[ \leq \frac{1}{50} E(0) + 1.2 \sqrt{\frac{14}{13}} \| \Delta x(0) \| \int_0^\infty e^{-t} dt \]
\[ \leq 1.672 = 0.532 \pi \text{ [radian]} \]

This shows that if the arm-hand system starts from the posture \( S \) of Fig.3 then the change of the third joint (wrist) angle does not beyond the value about \( \pi/2 \) [radian], that is, such a posture as indicated by \( U \) of Fig.3 where the angle of the wrist changes from \( \pi/6 \) to \(-5\pi/8\) will not arise. In the same way, it is possible to show that

\[ |q_i(\infty) - q_i(0)| \leq 0.222 \pi \text{ [radian]} \]

V. VARIABILITY AND OTHER CHARACTERISTICS OF MULTI-JOINT REACHING

As pointed out in the literature [30], the most noteworthy “reproducibility” of human skilled motion of reaching is “non-reproducibility” of joint motion trajectories. This is called “variability”, which is seen typically even in repeated motions played by professional athletes. However, variability of the endpoint in reaching is low relative to that of each joint motion. Let us show another simulation result when the following set of damping factors are chosen

\[ c_1 = c_2 = 0.69, \quad c_3 = c_4 = 0.096 \quad (30) \]

which makes the synergy of joint motions changed from balanced joint motions to harder non-universal joint motions (elbow and finger MP joint) and relatively softer universal joints (shoulder and wrist). Note that the endpoint trajectories \( \mathbf{x}(t) \) and \( \mathbf{x}(t) \) shown in Fig.10 are almost similar to those shown in Fig.6 and Fig.7 respectively. However, joint angle trajectories shown in Fig.11 became fairly different from those shown in Fig.8. Note that the shoulder angle \( q_1 \) moves from 70 [degree] to about 87 [degree] in Fig.11 but it moves from 70 [degree] to about 83.5 in Fig.8. The total movement of finger joint angle \( q_4 \) is reduced to 8 [degree] in Fig.11 from 18 [degree] in Fig.8.

In Fig.12 we show transient responses of endpoint velocities \( \dot{x}(t) \) and \( \dot{y}(t) \) in the case of damping factors of eq.(9). These two profiles are neither symmetric nor bell-shaped differently from the case of typical human skilled reaching. The reason is that in this paper each joint is assumed to be exerted directly from each corresponding actuator without taking actuator dynamics into consideration. Therefore, as seen in Fig.12, both profiles \( \dot{x}(t) \) and \( \dot{y}(t) \) rise up steeply immediately after \( t = 0 \) but converge to zero quite gently. However, if dynamics of generation of viscous-like forces exerted from an antagonist pair of muscles typical in shoulder and elbow joints are...
taken into consideration, then the convergence speed of those velocity profiles would be accelerated.

Finally we show the endpoint trajectory $x(t)$ of a global reaching movement in Fig.13 in which the initial position is set closely to a position with Jacobian singularity corresponding to $q_2 = q_3 = q_4 = 0$. Even in such a global reaching starting from a neighborhood of the Jacobian singularity, a desired reaching is attained by a proper set of damping coefficients like eq.(9), though the endpoint trajectory is not so close to the straight line.

VI. CONCLUSION

Human-like multi-joint reaching movements can be actualized by using a redundant DOF planar robot and a simple feedback scheme from task space to joint space with damping shaping without resolving inverse kinematics and introducing any artificial performance index. This result supports the EP hypothesis concerning human motor control proposed by Feldman [18] and Bizzi et al. [19] even in the case of excess-DOF reaching movements. At the same time, it is possible to claim that the proposed theoretical analysis and simulation results point to the importance of coordinated synergistic choice of effective damping factors among joints as well as that of “generation” of potential functions that exert spring-like forces on joints. Although this paper assumes that all joints can be directly exerted from actuators differently from the case of human reaching exerted by a group of muscles, the obtained results present some implications of importance of modeling of “viscous-like” dynamics of the neuro-muscular system effective as damping forces at the speed-down stage of reaching motion. This aspect has not yet been discussed not only in the literature of robotics research but also in the physiological literature except a series of papers by Hogan (for example, see Hogan [33]) and Gribble et al. [34], where change of mechanical impedance by coactivation of antagonist muscles is discussed [33] or enhancement of the feedback gain from spinal reflex loop by co-contraction of an antagonist pair of muscles is pointed out [34], which may result in increasing the viscosity in joints.

REFERENCES